MATH 579 Exam 1 Solutions

1. Choose 53 distinct integers in $[1, 100]$. Prove that two of them must differ by 13.

   Consider the 52 sets: $(1, 14), (2, 15), \ldots (13, 26), (27, 40), (28, 41), \ldots, (39, 52)$, $(53, 66), (54, 67), \ldots, (65, 78)$, $(79, 92), (80, 93), \ldots, (87, 100)$, $(88), (89), (90)$, (91). Each integer in $[1, 100]$ appears exactly once among these sets. By PHP, if we choose 53 distinct numbers, two must appear in the same set. However, those sets with two numbers have them differ by 13.

2. Choose 100 points in a unit square. Prove that seven of them must lie within some circle of radius 0.177.

   Divide the square as a checkerboard into 16 pieces. By the generalized PHP, some little square must have at least $\lceil \frac{100}{16} \rceil = 7$ points in it. The sides of the little square have length 0.25, so its diagonal has length $1/\sqrt{8}$. A circle of radius $\frac{1}{2\sqrt{8}} = 0.17678\ldots$ (or a larger one of radius .177) centered at the center of the little square will cover the whole little square.

3. A "lattice point" is a point (x, y) such that both x, y are integers. Prove that among any set of five lattice points there must be two lattice points whose midpoint is also a lattice point.

   The midpoint between (x_1, y_1) and (x_2, y_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$. Its first (resp. second) coordinate is an integer precisely when x_1, x_2 (resp. y_1, y_2) are of the same parity. Hence the natural bins to consider are (odd, odd), (odd, even), (even, odd), (even, even). Each lattice point falls into one of the bins. By PHP, if we choose five lattice points, two must be in the same bin; these two will have their midpoint a lattice point.

4. Prove that some (positive integer) power of 3 ends in 001. (i.e. $3^n = \ldots 001$).

   Consider $3^1, 3^2, \ldots, 3^{1001}$, and divide each of them by 1000. The remainders are among $0, 1, 2, \ldots, 999$; by PHP two remainders must be the same. Hence 1000 divides $3^i - 3^j = 3^j(3^{i-j} - 1)$, for some $1 \le j < i \le 1001$. But since neither 2 nor 5 divides 3^j, we must have 1000 dividing $3^{i-j} - 1$; that is, $3^{i-j} - 1 = 1000s$ for some positive integer s. But then $3^{i-j} = 1000s + 1$, which means that 3^{i-j} ends in 001.

5. Let $f(x)$ be a polynomial with integer coefficients. Suppose that $f(x)$ has nine different integer roots. Prove that, for every integer x, if $f(x) \in [-11, 11]$, then $f(x) = 0$.

   We begin with a lemma from class: If $g(x)$ is a polynomial with integer coefficients, and p, q are integers, then $p-q$ divides $g(p)-g(q)$. To prove this, we write the polynomial $g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and calculate $g(p) - g(q) = a_n(p^n - q^n) + a_{n-1}(p^{n-1} - q^{n-1}) + \cdots + a_1(p - q) + a_0(0)$. Because $p - q$ divides $p^k - q^k$ for every positive integer k [quick proof: $p^k - q^k = (p - q)(p^{k-1} + p^{k-2}q^1 + p^{k-3}q^2 + \cdots + p^1 q^{k-2} + q^{k-1})$], $p - q$ divides each summand, which proves the lemma.

Our second lemma is that every integer x in $[-11, 11]$, other than 0, has at most eight divisors. We prove this on a case-by-case basis. First, if x is prime, it has exactly four different integer divisors: $x, 1, -1, -x$. This handles $2, -2, 3, -3, 5, -5$, $7, -7, 11, -11$. If $x = \pm 1$, it has exactly two integer divisors: $1, -1$. If x is the product of two primes pq, it has exactly eight different integer divisors: $pq, p, q, 1, -1, -p, -q, -pq$. This handles $6, 10, -6, -10$. If x is the square of a prime p^2, it has exactly six different integer divisors: $p^2, p, 1, -1, -p, -p^2$. This handles $4, 9, -4, -9$. Finally, 8 has exactly eight different integer divisors: $8, 4, 2, 1, -1, -2, -4, -8$.

Now, let $r_1, r_2, \ldots, r_9$ be nine distinct integer roots of $f(x)$; hence $f(r_1) = f(r_2) = \cdots = f(r_9) = 0$. Suppose now that s is an integer with $f(s) \in [-11, 11]$. By the first lemma, $s - r_i$ divides $f(s) - f(r_i) = f(s)$. If $f(s) = 0$, the problem is solved, otherwise we will get a contradiction. By the second lemma, $f(s)$ has at most eight integer divisors, and there are nine integers $s - r_i$. By PHP two of them must be equal. Without loss of generality say $s - r_1 = s - r_2$. We rearrange and solve to get $r_1 = r_2$, which contradicts the assumption that $r_1, r_2, \ldots, r_9$ are distinct.

6. Let S be a set with n elements. Choose over half of the subsets of S; prove that two of the subsets you've chosen have one a subset of the other.

   Let x be an arbitrary element of S. We pair off the subsets of S, so that each pair will have one subset that contains x, and one that does not (and this is the ONLY difference between the two subsets). In other words, if $A, B, \ldots$ are all the subsets that do not contain x, then $(A, A \cup \{x\}), (B, B \cup \{x\}), \ldots$ are the pairs. For a more concrete example, consider $S = \{x, y, z\}$. Then our pairs are $(\{\}, \{x\}), (\{y\}, \{x, y\}), (\{z\}, \{x, z\}), (\{y, z\}, \{x, y, z\})$.

   Note that every subset of S appears exactly once in our pairing, and hence the number of pairs is exactly half of the number of subsets of S. If we choose *over* half of the subsets of S, by PHP two of the subsets chosen must be from the same pair. But our pairs were constructed so that one will always be a subset of the other.

Exam results: High score=88, Median score=66, Low score=52 (before any extra credit)